ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2017, Vol. 57, No. 3, pp. 387–400. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © W. Weigant, P.I. Plotnikov, 2017, published in Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 2017, Vol. 57, No. 3, pp. 382–395.

In memory of S.I. Pohozaev

Rotationally Symmetric Viscous Gas Flows

W. Weigant[a]* and P. I. Plotnikov[b, c]**

[a] *Bonn University, Bonn, Germany*
[b] *Novosibirsk State University, Novosibirsk, 630090 Russia*
[c] *Lavrent'ev Institute of Hydrodynamics, Siberian Branch, Russian Academy of Sciences, Novosibirsk, 630090 Russia*
*e-mail: *w-weigant@t-online.de; **plotnikov@hydro.nsc.ru*

Received July 26, 2016

Abstract—The Dirichlet boundary value problem for the Navier–Stokes equations of a barotropic viscous compressible fluid is considered. The flow region and the data of the problem are assumed to be invariant under rotations about a fixed axis. The existence of rotationally symmetric weak solutions for all adiabatic exponents from the interval (γ^*, ∞) with a critical exponent $\gamma^* < 4/3$ is proved.

Keywords: viscous gas, Navier–Stokes equations, rotational symmetry, Dirichlet boundary value problem, weak solutions.

DOI: 10.1134/S0965542517030150

1. FORMULATION OF THE PROBLEM AND THE MAIN RESULTS

We study an initial–boundary value problem for the three-dimensional equations of motion of a barotropic viscous gas. The gas is assumed occupy a bounded domain $\Omega \subset \mathbb{R}^3$. Its state is completely characterized by the density distribution $\varrho(x,t)$ and the velocity field $\mathbf{u}(x,t)$. The problem is to find $\mathbf{u}$ and ϱ satisfying the following equations and boundary and initial conditions in the cylinder $Q_T = \Omega \times (0,T)$:

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\varrho) = \operatorname{div}\mathbb{S}(\mathbf{u}) + \varrho\mathbf{f} \quad \text{in} \quad Q_T, \tag{1.1a}$$

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in} \quad Q_T, \tag{1.1b}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.1c}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) \quad \text{in} \quad \Omega. \tag{1.1d}$$

Here, $\mathbf{f}$ denotes a given mass force field and $\mathbb{S}(\mathbf{u})$ is the viscous stress tensor defined as

$$S(\mathbf{u}) = \nu_1(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + \nu_2\operatorname{div}\mathbf{u}\mathbb{I}, \tag{1.1e}$$

where the viscosity coefficients are constants satisfying the conditions $\nu_1 > 0$ and $\nu_1 + \nu_2 \geq 0$. It is assumed that $p = \varrho^\gamma$, where $\gamma > 1$ is the ratio of specific heats.

The first nonlocal results concerning the mathematical theory of boundary value problems for the two- and three-dimensional compressible Navier–Stokes equations were obtained by P.L. Lions. Specifically, he showed in [1] that the basic boundary value problems for these equations with a pressure function $p \sim \varrho^\gamma$ have weak solutions for all $\gamma > 5/3$ in the three-dimensional case and for all $\gamma > 3/2$ in the two-dimensional case. Later, the solvability of problem (1.1) for all $\gamma > 3/2$ in the three-dimensional case and for $\gamma > 1$ in the two-dimensional case was proved in [2]. For a detailed representation of this theory, the reader is referred to [3–5]. The solvability of problem (1.1) for $\gamma \leq 3/2$ remains an open question. The basic difficulty is associated with energy concentration (see [1, Chapter 6.6]). Specifically, a finite energy can concentrate in arbitrarily small domains, forming so-called concentrations. To avoid them, one has to prove that the total energy density of the gas is equipotentially absolutely integrable, i.e., has a better estimate than that in the $L^1(Q_T)$ norm. The problem simplifies considerably if the concentrations can be localized. This is possible if the flow has additional symmetry properties.

In this paper, we consider the case of a rotationally symmetric solution with possible concentrations located near the axis of rotation. To formulate the results more precisely, we introduce the following notation. Let τ, φ, and x_3 denote cylindrical coordinates in $\mathbb{R}^3$,

$$x_1 = \tau\cos\varphi, \quad x_2 = \tau\sin\varphi, \quad x_3 = x_3. \tag{1.2}$$

Let

$$\begin{aligned} u_\tau &= u_1\cos\varphi + u_2\sin\varphi, \quad u_\varphi = -u_1\sin\varphi + u_2\cos\varphi, \\ f_\tau &= f_1\cos\varphi + f_2\sin\varphi, \quad f_\varphi = -f_1\sin\phi + f_2\cos\varphi. \end{aligned} \tag{1.3}$$

Definition 1.1. A gas flow is said to be rotationally symmetric if $(\varrho, u_\tau, u_\varphi, u_3)$ are independent of φ.

Special cases of rotationally symmetric flows are axisymmetric flows with $u_\varphi = 0$ and spherically symmetric flows with $(\varrho, \mathbf{u})$ depending only on $|x|$. In the case of spherically symmetric flows, the existence of weak global solutions was proved for all $\gamma > 1$ in [6] (see also [7]). Axisymmetric and helically symmetric flows were studied in [8, 9], where the existence of weak solutions was also proved for all $\gamma > 1$. However, in contrast to the spherically symmetric case, the resulting solutions do not satisfy the equations on the axis of symmetry.

Our goal is to weaken the constraint $\gamma > 3/2$ for the case of rotationally symmetric flows and to prove the existence of weak solutions for all γ greater than some critical value $\gamma^* \in (1, 3/2)$. Hereafter, the flow region and the data of the problem are assumed to satisfy the following conditions.

Condition 1.1.

- $\Omega \subset \mathbb{R}^3$ is a bounded domain with a C^∞ boundary and is invariant under rotations about the x_3 axis.
- The functions ϱ_0, $u_{0,\tau}$, $u_{0,\varphi}$, $u_{0,3}$, f_τ, f_φ, and f_3 are independent of φ. Additionally, $\varrho_0, \mathbf{u}_0 \in L^\infty(\Omega)$, $\mathbf{f} \in L^\infty(Q_T)$, and

$$\|\mathbf{u}_0\|_{W_0^{1,2}(\Omega)} + \|\varrho_0\|_{L^\infty(\Omega)} + \|\mathbf{f}\|_{L^\infty(Q_T)} \le c, \quad \varrho_0 > c^{-1} > 0, \tag{1.4}$$

where c is a positive constant.

The weak solution of problem (1.1) is defined as follows.

Definition 1.2. A pair $\varrho \in L^\infty(0,T;L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega))$ is said to be a weak solution of problem (1.1) if the following conditions are satisfied:

- The kinetic energy of the flow is bounded, i.e., $\varrho|\mathbf{u}|^2 \in L^\infty(0,T;L^1(\Omega))$, and the gas density is nonnegative: $\varrho \ge 0$.
- The integral identity

$$\begin{aligned} &\int\limits_{Q_T} (\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\xi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\xi} + \varrho^\gamma \operatorname{div}\boldsymbol{\xi} - \mathbb{S}(\mathbf{u}) : \nabla\boldsymbol{\xi})dxdt \\ &\qquad + \int\limits_{Q_T} \varrho\mathbf{f}\cdot\boldsymbol{\xi}dxdt + \int\limits_\Omega \varrho_0\mathbf{u}_0(x)\cdot\boldsymbol{\xi}(x,0)dx = 0 \end{aligned} \tag{1.5}$$

holds for any vector field $\boldsymbol{\xi} \in C^\infty(Q_T)$ that vanishes in a neighborhood of $\partial\Omega\times[0,T]$ and $\Omega\times\{t = T\}$.
- The integral identity

$$\int\limits_{Q_T} (\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi)dxdt + \int\limits_\Omega \varrho_0(x)\psi(x,0)dx = 0 \tag{1.6}$$

holds for any function $\psi \in C^\infty(Q_T)$ that vanishes in a neighborhood of $\Omega\times\{t = T\}$.

Together with the basic problem (1.1), we consider the one-parameter family of regularized boundary value problems

$$\partial_t(\varrho_\varepsilon\mathbf{u}_\varepsilon) + \operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) + \nabla p_\varepsilon(\varrho_\varepsilon) = \operatorname{div}\mathbb{S}(\mathbf{u}_\varepsilon) + \varrho_\varepsilon\mathbf{f} \quad \text{in} \quad Q_T, \tag{1.7a}$$

$$\partial_t\varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon) = 0 \quad \text{in} \quad Q_T, \tag{1.7b}$$

$$\mathbf{u}_\varepsilon = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.7c}$$

$$\mathbf{u}_\varepsilon(x,0) = \mathbf{u}_0(x), \quad \varrho_\varepsilon(x,0) = \varrho_0(x) \quad \text{in} \quad \Omega, \tag{1.7d}$$

where the modified pressure function is given by

$$p_\varepsilon(\varrho) = \varrho^\gamma + \varepsilon\varrho^m, \quad \varepsilon \in (0,1], \quad m \geq 6. \tag{1.7e}$$

The existence of a weak solution to problem (1.7) was established in [3, 1]. The following proposition is a consequence of those results.

Proposition 1.1. *Let the domain* Ω *and the functions* $\mathbf{u}_0$, ϱ_0, *and* $\mathbf{f}$ *satisfy Conditions* 1.1. *Then problem* (1.7) *has a rotationally symmetric weak solution* $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ *obeying the following conditions*:

(i) *The functions* $\varrho_\varepsilon \geq 0$ *and* $\mathbf{u}_\varepsilon$ *satisfy the energy inequality*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \{\varrho_\varepsilon |\mathbf{u}_\varepsilon^2| + \varrho_\varepsilon^\gamma + \varepsilon\varrho_\varepsilon^m)dx + \int_{Q_T} |\nabla \mathbf{u}_\varepsilon|^2 dxdt \leq c, \tag{1.8}$$

where c *is independent of* ε.

(ii) *The integral identity*

$$\int_{Q_T} (\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\xi})dxdt + \int_{Q_T} (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon + p_\varepsilon(\varrho_\varepsilon)\mathbb{I} - \mathbb{S}(\mathbf{u}_\varepsilon)) : \nabla\boldsymbol{\xi} dxdt$$
$$+ \int_{Q_T} \varrho_\varepsilon \mathbf{f} \cdot \boldsymbol{\xi} dxdt + \int_\Omega \varrho_0(x)\mathbf{u}_0(x) \cdot \boldsymbol{\xi}(x,0)dx = 0 \tag{1.9}$$

holds for all vector fields $\boldsymbol{\xi} \in C^\infty(Q)$ *satisfying the conditions*

$$\boldsymbol{\xi}(x,T) = 0 \quad in \quad \Omega, \quad \boldsymbol{\xi}(x,t) = 0 \quad on \quad \partial\Omega \times (0,T).$$

(iii) *The integral identity*

$$\int_{Q_T} (\varrho_\varepsilon \partial_t \psi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\psi)dxdt + \int_\Omega \psi(x,0)\varrho_0 dx = 0 \tag{1.10}$$

holds for all smooth functions ψ *vanishing in a neighborhood of* $\Omega \times \{t = T\}$.

Passing to a subsequence, we can assume that there exist $(\varrho, \mathbf{u})$ such that

$$\varrho_\varepsilon \rightharpoonup \varrho \quad \text{star-weakly in} \quad L^\infty(0,T;L^\gamma(\Omega)),$$
$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in} \quad L^2(0,T;W_0^{1,2}(\Omega)). \tag{1.11}$$

Obviously, the limiting functions also satisfy the energy estimate

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \{\varrho|\mathbf{u}^2| + \varrho^\gamma)dx + \int_{Q_T} |\nabla\mathbf{u}|^2 dxdt \leq c. \tag{1.12}$$

To prove the solvability of problem (1.1), it is sufficient to show that each term in integral identities (1.9) and (1.10) converges as $\varepsilon \to 0$ to the corresponding term in integral identities (1.5) and (1.6). According to the general theory of the viscous compressible Navier–Stokes equations (see [4, 5]), the passage to the limit will be substantiated if we prove that the energy density on any compact subset of Q_T satisfies an estimate stronger than the energy one. More precisely, it is sufficient to show that, for any compact set $K \Subset Q_T$,

$$\int_K (\varrho_\varepsilon^{\gamma+\vartheta} + \varepsilon\varrho_\varepsilon^{\gamma+\vartheta})dxdt \leq c, \tag{1.13}$$

$$\int_K (\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2)^{1+\vartheta} dxdt \leq c, \tag{1.14}$$

where $\vartheta > 0$ is a constant independent of K or ε, while c is a constant depending on K, but independent of ε. It is well known (see [4, 5]) that estimate (1.13) is a consequence of estimate (1.14) and Bogovskii's lemma. Therefore, the solvability of problem (1.1) is reduced to the validity of inequality (1.14) for all

$\gamma > \gamma^*$. The determination of the minimum possible critical exponent $\gamma^* \in (1, 3/2)$ is an integral part of the problem. The solution of this issue is given by the following theorem, which is the main result of this paper.

Theorem 1.3. *Let Conditions* 1.1 *hold. Then, for any*

$$\gamma > \gamma^* = (7 + \sqrt{73})/12$$

and $K \Subset Q_T$,

$$\left\| \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \right\|_{L^{1+\vartheta}(K)} \le c,$$

where $c > 0$ *and* $\vartheta > 0$ *are independent of* ε.

This theorem implies that, for $\gamma > \gamma^*$, problem (1.1) has a rotationally symmetric weak solution satisfying all the conditions of Definition 1.2. It should be noted that the condition $\gamma > \gamma^*$ is satisfied by $\gamma = 4/3$ (polyatomic gases), $\gamma = 7/5$ (diatomic gases), and $\gamma = 5/3$ (monatomic gases). The existence of weak solutions for all $\gamma > 1$ remains an open question. The rest of this paper deals with the proof of Theorem 1.3.

2. LOCALIZATION AND ESTIMATES FOR POTENTIALS

2.1. Localization

The first step in the proof of Theorem 1.3 is to localize the equations in a neighborhood of an arbitrary compact set $K \Subset Q_T$. The localization procedure is based on the following auxiliary result.

Lemma 2.1. *For any compact set* $K \Subset Q_T$, *there are functions* $\zeta, \eta \in C_0^\infty(Q_T)$ *with the properties*

$$\zeta = \eta = 1 \quad on \quad K, \quad \eta = 1 \quad on \quad \operatorname{supp}\zeta, \quad 0 \le \zeta, \quad \eta \le 1, \tag{2.1}$$

and ζ *and* η *are independent of the angular variable* φ.

Proof. Let $\varsigma \in C^\infty(\mathbb{R}^3)$ be an arbitrary fixed monotone function such that $\varsigma(s) = 0$ for $s < 1/2$ and $\varsigma(s) = 1$ for $s > 1$. Let

$$v_d(x,t) = \varsigma(d^{-1}\operatorname{dist}(x, \partial\Omega))\varsigma(d^{-1}\operatorname{dist}(t, \{0, T\})).$$

Define

$$d = \operatorname{dist}(K, \partial Q_T), \quad \eta^*(x,t) = v_{d/6}(x,t), \quad \zeta^*(x,t) = v_{d/2}(x,t).$$

The functions ζ^* and η^* are extended by zero to $\mathbb{R}^3$. The resulting functions are continuous and satisfy the inequalities $0 \le \zeta^*$ and $\eta^* \le 1$. Moreover, $\zeta^* = 1$ in the $d/2$-neighborhood of the compact set K, $\eta^* = 1$ in the $d/3$-neighborhood of the support of ζ^*, and $\zeta^* = \eta^* = 0$ in the $d/12$-neighborhood of ∂Q_T. Since Q_T is rotationally symmetric about the x_3 axis, the function ζ^* is also rotationally symmetric. For any $\epsilon > 0$, we define the averaging operator

$$\mathcal{L}_\epsilon f(x,t) := \frac{1}{\epsilon^4}\int_{\mathbb{R}^4} \omega(\epsilon^{-1}|x-y|)\omega(\epsilon^{-1}(t-s)) f(y,s)\,dyds,$$

where the standard averaging kernel ω is a nonnegative even infinitely differentiable function compactly supported in the interval $[-1, 1]$. Its integral over the number line is equal to 1. Obviously, for sufficiently small $\epsilon > 0$, the functions $\zeta = \mathcal{L}_\epsilon \zeta^*$ and $\eta = \mathcal{L}_\epsilon \eta^*$ satisfy all the assumptions of the lemma.

Consider the functions

$$\rho_\varepsilon = \zeta \varrho_\varepsilon, \quad \mathbf{v}_\varepsilon = \eta \mathbf{u}_\varepsilon. \tag{2.2}$$

Assume that they are continued by zero outside Q_T. The properties of these functions are described by the following assertion.

Lemma 2.2. *There is* $R > 0$ *such that* $\rho_\varepsilon(x,t) = 0$ *and* $\mathbf{v}_\varepsilon(x,t) = 0$ *for* $|x| + |t| \ge R$. *Moreover,*

$$\|\rho_\varepsilon\|_{L^\infty(\mathbb{R}^1; L^\gamma(\mathbb{R}^3))} + \left\|\rho_\varepsilon |\mathbf{v}_\varepsilon|^2\right\|_{L^\infty(\mathbb{R}^1; L^1(\mathbb{R}^3))} + \|\mathbf{v}_\varepsilon\|_{L^2(\mathbb{R}^1; W_0^{1,2}(\mathbb{R}^3))} \le c(K). \tag{2.3}$$

Here, $c(K)$ is a constant depending on K, but independent of ε. Moreover, for any $\xi \in C^\infty(\mathbb{R}^4)$, the functions $(\rho_\varepsilon, \mathbf{v}_\varepsilon)$ satisfy the integral identity

$$\int_{\mathbb{R}^4} (\rho_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \boldsymbol{\xi}) dxdt + \int_{\mathbb{R}^4} (\rho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon + P_\varepsilon \mathbb{I} - \mathbb{T}_\varepsilon) : \nabla \boldsymbol{\xi} dxdt + \int_{\mathbb{R}^4} \mathbf{F}_\varepsilon \cdot \boldsymbol{\xi} dxdt = 0, \tag{2.4}$$

where the functions P_ε, $\mathbb{T}_\varepsilon$, and $\mathbf{F}_\varepsilon$ vanish for $|x| + |t| \geq R$ and satisfy the estimates

$$\begin{gathered} \|\mathbb{T}_\varepsilon\|_{L^2(\mathbb{R}^4)} + \|\mathbf{F}_\varepsilon\|_{L^1(\mathbb{T}_\varepsilon{}^4)} \leq c, \\ \rho_\varepsilon^\gamma + \varepsilon \rho_\varepsilon^m \leq P_\varepsilon. \end{gathered} \tag{2.5}$$

Proof. The compactness of the support of $(\rho_\varepsilon, \mathbf{v}_\varepsilon)$ follows from (2.2). Estimate (2.3) is a consequence of (2.2) and energy estimate (1.8). To derive integral identity (2.4), we choose an arbitrary $\boldsymbol{\xi} \in C^\infty(\mathbb{R}^4)$ and replace $\boldsymbol{\xi}$ by $\zeta\boldsymbol{\xi}$ in integral identity (1.9). As a result,

$$\begin{aligned} &\int_{\mathbb{R}^4} (\rho_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \boldsymbol{\xi}) dxdt + \int_{\mathbb{R}^4} (\rho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon + \zeta p_\varepsilon(\varrho_\varepsilon) \mathbb{I} - \zeta \mathbb{S}(\mathbf{u}_\varepsilon)) : \nabla \boldsymbol{\xi} dxdt \\ &+ \int_{\mathbb{R}^4} (\rho_\varepsilon \mathbf{f} + (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \nabla \zeta + \varrho_\varepsilon \mathbf{u}_\varepsilon \partial_t \zeta - \mathbb{S}(\mathbf{u}_\varepsilon) \nabla \zeta) \cdot \boldsymbol{\xi} dxdt = 0. \end{aligned} \tag{2.6}$$

Defining

$$\begin{gathered} \mathbf{F}_\varepsilon = \rho_\varepsilon \mathbf{f} + (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \nabla \zeta + \varrho_\varepsilon \mathbf{u}_\varepsilon \partial_t \zeta - \mathbb{S}(\mathbf{u}_\varepsilon) \nabla \zeta, \\ P_\varepsilon = \zeta(\varrho^\gamma + \varepsilon \varrho^m), \quad \mathbb{T}_\varepsilon = \zeta \mathbb{S}(\mathbf{u}_\varepsilon). \end{gathered}$$

and substituting these expressions into (2.6), we obtain (2.4). Estimates (2.5) are a straightforward consequence of energy estimate (1.8).

2.2. Estimates of Potentials

In this section, we prove the basic estimates for the potentials of the functions ρ_ε. These estimates will play a key role in the proof of Theorem 1.3. Consider a one-parameter family of integrals

$$\mathbf{G}(r) = \int_{\mathbb{R}^4} \rho_\varepsilon^\gamma(x,t) \left\{ \int_{|x-y| \leq \mathbf{r}} |x-y|^{-1} \rho_\varepsilon(y,t) dy \right\} dxdt. \tag{2.7}$$

Proposition 2.1. *Under the conditions of Lemma* 2.2, *for any $r \in (0, 1/2)$,*

$$\mathbf{G}(r) \leq c r^{2(\gamma-1)/\gamma}, \tag{2.8}$$

where c is a constant independent of ε or r.

The proof relies on the following lemma.

Lemma 2.3. *Under the conditions of Lemma* 2.2, *satisfies the estimate*

$$\int_{\mathbb{R}^4} |x|^{-1} \rho_\varepsilon^\gamma dxdt \leq c, \tag{2.9}$$

where the constant c is independent of ε.

Proof. For any $\sigma > 0$, let

$$\boldsymbol{\xi}_\sigma = |x|^{-1} x \quad \text{if} \quad |x| \geq \sigma, \quad \boldsymbol{\xi}_\sigma = \sigma^{-1} x \quad \text{if} \quad |x| \leq \sigma.$$

Substituting $\boldsymbol{\xi}_\sigma$ into integral identity (2.4) yields the relation

$$\int_{\mathbb{R}^4} \rho_\varepsilon v_{\varepsilon,i} v_{\varepsilon,j} Q_{ij}(x) dxdt + \int_{\mathbb{R}^4} P_\varepsilon \sum_i Q_{ii}(x) dxdt - \int_{\mathbb{R}^4} T_{\varepsilon,ij} Q_{ij}(x) dxdt + \int_{\mathbb{R}^4} \mathbf{F}_\varepsilon \cdot \boldsymbol{\xi}_\sigma dxdt = 0, \tag{2.10}$$

where

$$Q_{ij} = -|x|^{-3} x_i x_j \quad \text{if} \quad j \neq j, \quad |x| \geq \sigma,$$
$$Q_{ii} = |x|^{-3} (|x|^2 - x_i^2) \quad \text{if} \quad |x| \geq \sigma,$$
$$Q_{ij} = \sigma^{-1}\delta_{ij} \quad \text{if} \quad |x| \leq \sigma.$$

From this, we find

$$\rho_\varepsilon v_{\varepsilon,i} v_{\varepsilon,j} Q_{ij} = \rho_\varepsilon |x|^{-3} (|x|^2 |\mathbf{v}_\varepsilon|^2 - (\mathbf{v}_\varepsilon \cdot x)^2) \geq 0 \quad \text{if} \quad |x| \geq \sigma,$$
$$P_\varepsilon \sum_i Q_{ii} = 2|x|^{-1} P_\varepsilon \quad \text{if} \quad |x| \geq \sigma,$$
$$\rho_\varepsilon v_{\varepsilon,i} v_{\varepsilon,j} Q_{ij} = \rho_\varepsilon |\mathbf{v}|^2 \sigma^{-1}, \quad P_\varepsilon \sum_i Q_{ii} = 3\sigma^{-1} P_\varepsilon \quad \text{if} \quad |x| \leq \sigma.$$

Combining these relations with (2.10) and (2.5) gives the estimate

$$\int_{\mathbb{R}} \int_{|x|\geq\sigma} |x|^{-1} \rho_\varepsilon^\gamma dxdt + \frac{3}{\sigma} \int_{\mathbb{R}} \int_{|x|\leq\sigma} \rho_\varepsilon^\gamma dxdt \leq c \int_{\mathbb{R}^4} |x|^{-1} |\mathbb{T}_\varepsilon| dxdt + c \int_{\mathbb{R}^4} |\mathbf{F}_\varepsilon| dxdt. \tag{2.11}$$

Recall that the functions $\mathbb{T}_\varepsilon$ and $\mathbf{F}_\varepsilon$ are supported by the compact set $|x| + |t| \leq R$. Furthermore, for almost every $t \in (0, T)$, we have

$$\int_{\mathbb{R}^3} |x|^{-1} |\mathbb{T}_\varepsilon(x,t)| \, dx \leq c \|\mathbb{T}_\varepsilon(t)\|_{L^2(\mathbb{R}^3)}.$$

Applying estimates (2.5) yields

$$\int_{\mathbb{R}^4} |x|^{-1} |\mathbb{T}_\varepsilon| dxdt + \int_{\mathbb{R}^4} |\mathbf{F}_\varepsilon| dxdt \leq c.$$

Substituting these inequalities into (2.11), we find that

$$\int_{\mathbb{R}} \int_{|x|\geq\sigma} |x|^{-1} \rho_\varepsilon^\gamma dxdt \leq c.$$

Sending σ to zero produces the desired estimate (2.9).

Let us return to the proof of Proposition 2.1. For every $x \in \mathbb{R}^3$, let

$$D_1(x) = \left\{ y \in \mathbb{R}^3 : |x - y| \leq r, \, |x - y| \leq 2^{-1}|x| \right\},$$
$$D_2(x) = \left\{ y \in \mathbb{R}^3 : |x - y| \leq r, \, |x - y| \geq 2^{-1}|x| \right\}.$$

It holds that

$$\mathbf{G}(r) = \mathbf{G}_1(r) + \mathbf{G}_2(r), \quad \text{where} \quad \mathbf{G}_i(r) = \int_{\mathbb{R}^4} \rho_\varepsilon^\gamma(x,t) \left\{ \int_{D_i(x)} |x - y|^{-1} \rho_\varepsilon(y,t) dy \right\} dxdt. \tag{2.12}$$

Let us estimate $\mathbf{G}_1$. Note that

$$\int_{D_1(x)} |x - y|^{-1} \rho_\varepsilon(y,t) dy = \int_0^r s^{-2} \left\{ \int_{B(x,s)\cap D_1(x)} \rho_\varepsilon dy \right\} ds + r^{-1} \int_{B(x,\mathbf{r})\cap D_1(x)} \rho_\varepsilon dy. \tag{2.13}$$

Let $\Pi_\alpha \subset \mathbb{R}^3$ denote the sector

$$\Pi_\alpha = \{ x = (x_1, x_2, x_3) : |\varphi| \leq \alpha \},$$

where $x_1 + ix_2 = \tau e^{i\varphi}$. Since ρ_ε is rotationally symmetric, we have

$$\int_{\Pi_\alpha} \rho_\varepsilon^\gamma(y,t)dy = \pi^{-1}\alpha\int_{\mathbb{R}^3} \rho_\varepsilon^\gamma(y,t)dy \le c\alpha. \tag{2.14}$$

This inequality is used to estimate the integral over the ball $B(x,s)$ on the right-hand side of (2.13). Since $\rho_\varepsilon(x,t)$ is rotationally symmetric, it is sufficient to consider the case $x = (x_1, 0, x_3)$. In this case, we have the inclusion

$$D_1(x) \cap B(x,s) \subset \Pi_\alpha, \quad \alpha = \arcsin(|x|^{-1}s),$$

which, together with (2.14), yields the estimate

$$\int_{B(x,s)\cap D_1(x)} \rho_\varepsilon dy \le \left(\int_{B(x,s)} dy\right)^{(\gamma-1)/\gamma}\left(\int_{\Pi_\alpha}\rho_\varepsilon^\gamma(y,t)dy\right)^{1/\gamma}$$
$$\le cs^{3(\gamma-1)/\gamma}(|x|^{-1}s)^{1/\gamma} = cs^{(3\gamma-2)/\gamma}|x|^{-1/\gamma}, \quad (3\gamma-2)/\gamma > 1.$$

It follows that

$$\int_0^r s^{-2}\left\{\int_{B(x,s)\cap D_1(x)} \rho_\varepsilon dy\right\}ds \le c|x|^{-1/\gamma}r^{(3\gamma-2)/\gamma-1} = c|x|^{-1/\gamma}r^{2(\gamma-1)/\gamma}.$$

Repeating the above argument, we find

$$\int_{B(x,\mathbf{r})\cap D_1(x)} \rho_\varepsilon dy \le \left(\int_{B(x,\mathbf{r})} dy\right)^{(\gamma-1)/\gamma}\left(\int_{\Pi_\alpha}\rho_\varepsilon^\gamma(y,t)dy\right)^{1/\gamma}$$
$$\le cr^{3(\gamma-1)/\gamma}(|x|^{-1}r)^{1/\gamma} = cr^{(3\gamma-2)/\gamma}|x|^{-1/\gamma}, \quad (3\gamma-2)/\gamma > 1.$$

Substituting the resulting estimates into (2.13) yields

$$\int_{D_1(x)} |x-y|^{-1}\rho_\varepsilon(y,t)dy \le c|x|^{-1/\gamma}r^{2(\gamma-1)/\gamma}.$$

Combining this result with inequality (2.9), we derive the desired estimate for $\mathbf{G}_1$:

$$\begin{aligned}\mathbf{G}_1(r) &= \int_{\mathbb{R}^4}\rho_\varepsilon^\gamma(x,t)\left\{\int_{D_1(x)}|x-y|^{-1}\rho_\varepsilon(y,t)dy\right\}dxdt \\ &\le cr^{2(\gamma-1)/\gamma}\int_{\mathbb{R}^4}|x|^{-1/\gamma}\rho_\varepsilon^\gamma(x,t)dxdt \le cr^{2(\gamma-1)/\gamma}.\end{aligned} \tag{2.15}$$

To estimate $\mathbf{G}_2$, we note that the definition of the set $D_2(x)$ implies the inequality

$$\int_{D_2(x)} |x-y|^{-1}\rho_\varepsilon(y,t)dy \le c|x|^{-1}\int_{|x-y|\le\mathbf{r}}\rho_\varepsilon(y,t)dy \le c|x|^{-1}\left(\int_{|x-y|\le\mathbf{r}}dy\right)^{(\gamma-1)/\gamma}$$
$$\times\left(\int_{|x-y|\le\mathbf{r}}\rho_\varepsilon^\gamma(y,t)dy\right)^{1/\gamma} \le c|x|^{-1}r^{3(\gamma-1)/\gamma}.$$

Substituting this inequality into expression (2.12) for $\mathbf{G}_2$ and applying estimate (2.9), we finally obtain

$$\mathbf{G}_2 \le cr^{3(\gamma-1)/\gamma}. \tag{2.16}$$

It remains to be noted that estimate (2.8) is an obvious consequence of (2.15) and (2.16).

The result below follows directly from Proposition 2.1.

Proposition 2.2. *Under the conditions of Proposition* 2.1, *we have the estimate*

$$\int_{\mathbb{R}^4}\left(\int_{|x-y|\le \mathbf{r}}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy\right)^2 dxdt \le cr^{2(\gamma-1)/\gamma}, \tag{2.17}$$

where c *is a constant independent of* ε *or* $r \in (0, 1/2)$.

Proof. For any $\lambda \in \mathbb{R}$ and $r > 0$, let

$$|x|_{\mathbf{r}}^{\lambda} = |x|^{\lambda} \quad \text{if} \quad |x| \le r; \qquad |x|_{\mathbf{r}}^{\lambda} = 0 \quad \text{if} \quad |x| > r.$$

Expression (2.7) for $\mathbf{G}$ implies the representation

$$2\mathbf{G}(r) = \int_{\mathbb{R}^7}|x-y|_{\mathbf{r}}^{-1}(\rho_\varepsilon^\gamma(x,t)\rho_\varepsilon(y,t) + \rho_\varepsilon^\gamma(y,t)\rho_\varepsilon(x,t))dxdydt. \tag{2.18}$$

Note that

$$\rho_\varepsilon^{(\gamma+1)/2}(x,t)\rho_\varepsilon^{(\gamma+1)/2}(y,t) = (\rho_\varepsilon(x,t)\rho_\varepsilon(y,t))(\rho_\varepsilon^{(\gamma-1)/2}(x,t)\rho_\varepsilon^{(\gamma-1)/2}(y,t))$$
$$\le \frac{1}{2}(\rho_\varepsilon(x,t)\rho_\varepsilon(y,t))(\rho_\varepsilon^{\gamma-1}(x,t) + \rho_\varepsilon^{\gamma-1}(y,t)) = \frac{1}{2}(\rho_\varepsilon^\gamma(x,t)\rho_\varepsilon(y,t) + \rho_\varepsilon^\gamma(y,t)\rho_\varepsilon(x,t)).$$

Substituting this inequality into (2.18), we obtain the estimate

$$\int_{\mathbb{R}^7}|x-y|_{\mathbf{r}}^{-1}\rho_\varepsilon^{(\gamma+1)/2}(x,t)\rho_\varepsilon^{(\gamma+1)/2}(y,t)dxdydt \le \mathbf{G}(r).$$

Combining this inequality with (2.8) yields

$$\int_{\mathbb{R}^7}|x-y|_{2\mathbf{r}}^{-1}\rho_\varepsilon^{(\gamma+1)/2}(x,t)\rho_\varepsilon^{(\gamma+1)/2}(y,t)dxdydt \le \mathbf{G}(2r) \le cr^{2(\gamma-1)/\gamma}. \tag{2.19}$$

The standard estimates for the convolution of potentials show that

$$\int_{R^3}|x-z|_r^{-2}|y-z|_r^{-2}dz \le c|x-y||_{2r}^{-1},$$

which, combined with (2.19), yields the inequality

$$\int_{\mathbb{R}^4}\left\{\int_{R^3}|x-z|_{\mathbf{r}}^{-2}\rho_\varepsilon^{(\gamma+1)/2}(x,t)dx\right\}\left\{\int_{\mathbb{R}^3}|y-z|_{\mathbf{r}}^{-2}\rho_\varepsilon^{(\gamma+1)/2}(y,t)dy\right\}dzdt$$
$$\le \int_{\mathbb{R}^7}|x-y|_{2\mathbf{r}}^{-1}\rho_\varepsilon^{(\gamma+1)/2}(x,t)\rho_\varepsilon^{(\gamma+1)/2}(y,t)dxdydt \le \mathbf{G}(2r) \le cr^{2(\gamma-1)/\gamma}.$$

To derive estimate (2.17), it remains to be noted that

$$\int_{\mathbb{R}^4}\left\{\int_{\mathbb{R}^3}|x-z|_{\mathbf{r}}^{-2}\rho_\varepsilon^{(\gamma+1)/2}(x,t)dx\right\}\left\{\int_{\mathbb{R}^3}|y-z|_{\mathbf{r}}^{-2}\rho_\varepsilon^{(\gamma+1)/2}(y,t)dy\right\}dzdt$$
$$= \int_{\mathbb{R}^4}\left(\int_{|x-y|\le\mathbf{r}}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy\right)^2 dxdt.$$

Corollary 2.1. Under the conditions of Proposition 2.1, we have the estimate

$$\int_{\mathbb{R}^4}\left(\int_{\mathbb{R}^3}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy\right)^2 dxdt \le c, \tag{2.20}$$

where c is a constant independent of ε.

Proof. It is true that

$$\int_{\mathbb{R}^4}\left(\int_{\mathbb{R}^3}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy\right)^2dxdt=\int_{\mathbb{R}^4}F_1(x,t)^2dxdt+\int_{\mathbb{R}^4}F_2(x,t)^2dxdt, \tag{2.21}$$

where

$$F_1(x,t)=\int_{|x-y|\le 1/2}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy,\quad F_2(x,t)=\int_{|x-y|\ge 1/2}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy.$$

Proposition 2.2 implies the estimate

$$\int_{\mathbb{R}^4}F_1(x,t)^2dxdt\le c. \tag{2.22}$$

Since ρ_ε is supported by the ball $|x|+|t|\le R$ and $\|\rho_\varepsilon^\gamma(t)\|_{L^1(\mathbb{R}^3)}\le c$, we have

$$|F_2(x,t)|\le c(1+|x|)^{-2}.$$

Moreover, the function F_2 is supported by the layer $|t|\le R$. It follows that

$$\int_{\mathbb{R}^4}F_2(x,t)^2dxdt\le c. \tag{2.23}$$

It remains to be noted that the desired estimate (2.20) is a direct consequence of (2.22) and (2.23).

3. PROOF OF THEOREM 1.3

3.1. Estimates for the Density in Sobolev Spaces

In this section, we apply Proposition 2.2 in order to estimate $\rho_\varepsilon^{(\gamma+1)/2}$ in a negative Sobolev space. Recall that, for any $s\in\mathbb{R}$, the Hilbert space $H^s(\mathbb{R}^3)$ consists of all slowly growing generalized functions $u\in\mathcal{D}'(\mathbb{R}^3)$ with a finite norm

$$\|u\|_s=\left(\int_{\mathbb{R}^3}(1+|\xi|)^{2s}|\mathcal{F}u(\xi)|^2\,d\xi\right)^{1/2},$$

where $\mathcal{F}u(\xi)$ is the Fourier transform of a generalized function u. The following proposition is the main result of this section.

Proposition 3.1. *For any* $\lambda>(\gamma+1)/(3\gamma-1)$,

$$\|\rho_\varepsilon^{(\gamma+1)/2}\|_{L^2(\mathbb{R};H^{-\lambda}(\mathbb{R}^3))}\le c, \tag{3.1}$$

where c is independent of ε.

Proof. Consider an arbitrary function $u\in C_0^\infty(\mathbb{R}^4)$ supported by the ball $|x|+|t|\le 2R$. Let $w=(-\Delta)^{1/2}u$. It is well known that

$$u(x,t)=c_3\int_{\mathbb{R}^3}|x-y|^{-2}w(y,t)dy,\quad w\in C^\infty(\mathbb{R}^4)\cap L^2(\mathbb{R}^4), \tag{3.2}$$

where c_3 is an absolute constant. Then

$$\int_{\mathbb{R}^4}\rho_\varepsilon^{(\gamma+1)/2}udxdt=c_3\int_{\mathbb{R}^4}w(x,t)\left(\int_{\mathbb{R}^3}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy\right)dxdt. \tag{3.3}$$

Fix an arbitrary monotone function $\varsigma\in C^\infty(\mathbb{R})$ such that $\varsigma(s)=0$ if $s<1/2$ and $\varsigma(s)=1$ if $s\ge 1$. Let

$$K_1(|x|)=c_3\varsigma(|x|/r)|x|^{-2},\quad K_2(x)=c_3(1-\varsigma(|x|/r))|x|^{-2}.$$

It follows from (3.3) that

$$\int\limits_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} u\,dxdt = \int\limits_{\mathbb{R}^4} w(x,t)\left(\int\limits_{\mathbb{R}^3} K_1(x-y)\rho_\varepsilon^{(\gamma+1)/2}(y,t)dy\right)dxdt + \int\limits_{\mathbb{R}^4} w(x,t)\left(\int\limits_{\mathbb{R}^3} K_2(x-y)\rho_\varepsilon^{(\gamma+1)/2}dy\right)dxdt. \tag{3.4}$$

Introducing the function $g = (1-\Delta)^{-1}w$, we transform the first integral on the right-hand side of (3.4) into the form

$$\int\limits_{\mathbb{R}^4} w(x,t)\left(\int\limits_{\mathbb{R}^4} K_1(x-y)\rho_\varepsilon^{(\gamma+1)/2}dy\right)dxdt = \int\limits_{\mathbb{R}^4} g(x,t)(1-\Delta)(\int\limits_{\mathbb{R}^4} K_1(x-y)\rho_\varepsilon^{(\gamma+1)/2}dy)dxdt.$$

Note that K_1 satisfies the inequality

$$|(1-\Delta)K_1(x-y)| \le cr^{-2}|x-y|^{-2},$$

which implies the estimate

$$\left|\int\limits_{\mathbb{R}^4} g(x,t)(1-\Delta)\left(\int\limits_{\mathbb{R}^3} K_1(x-y)\rho_\varepsilon^{(\gamma+1)/2}dy\right)dxdt\right| \le cr^{-2}\int\limits_{\mathbb{R}^4}|g(x,t)|\left(\int\limits_{\mathbb{R}^3}|x-y|^{-2}\rho_\varepsilon^{(\gamma+1)/2}dy\right)dxdt.$$

Combining this with (2.20), we find

$$\left|\int\limits_{\mathbb{R}^4} g(x,t)(1-\Delta)\left(\int\limits_{\mathbb{R}^3} K_1(x-y)\rho_\varepsilon^{(\gamma+1)/2}dy\right)dxdt\right| \le cr^{-2}\|g\|_{L^2(\mathbb{R}^4)}. \tag{3.5}$$

Note that $|K_2(x)| \le |x|^{-2}$ and $K_2(x) = 0$ for $|x| \ge r$. Combining this result with (2.17) yields

$$\left|\int\limits_{\mathbb{R}^4} w(x,t)\left(\int\limits_{\mathbb{R}^3} K_2(x-y)\rho_\varepsilon^{(\gamma+1)/2}dy\right)dxdt\right| \le cr^{(\gamma-1)/\gamma}\|w\|_{L^2(\mathbb{R}^4)},$$

which, together with (3.5) and (3.4), gives the estimate

$$\left|\int\limits_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/\gamma} u\,dxdt\right| \le cr^{(\gamma-1)/\gamma}\|w\|_{L^2(\mathbb{R}^4)} + cr^{-2}\|g\|_{L^2(\mathbb{R}^4)}. \tag{3.6}$$

By the definition of the space $H^s(\mathbb{R}^3)$, we have

$$\|w\|_{L^2(\mathbb{R}^4)} \le c\|u\|_{L^2(\mathbb{R};H^1(\mathbb{R}^3))}, \quad \|g\|_{L^2(\mathbb{R}^4)} = \|(1-\Delta)^{-1}(-\Delta)^{1/2}u\|_{L^2(\mathbb{R}^4)} \le c\|u\|_{L^2(\mathbb{R};H^{-1}(\mathbb{R}^3))}.$$

From this relation and (3.6), it follows that any function $u \in C_0^\infty(\mathbb{R}^4)$ supported by the ball $|x| + |t| \le 2R$ satisfies the estimate

$$\left|\int\limits_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} u\,dxdt\right| \le cr^{(\gamma-1)/2}\|u\|_{L^2(\mathbb{R};(H^1(\mathbb{R}^3))} + cr^{-2}\|u\|_{L^2(\mathbb{R};H^{-1}(\mathbb{R}^3))}. \tag{3.7}$$

Let $h \in C^\infty(\mathbb{R}^4)$ be an arbitrary function equal to 1 in the ball $|x| + |t| \le R$ and vanishing outside the ball $|x| + |t| \le 2R$. Obviously, for any $v \in C_0^\infty(\mathbb{R}^4)$, the function $u = hv$ is supported by the ball $|x| + |t| \le 2R$ and satisfies the inequalities

$$\|u\|_{L^2(\mathbb{R};(H^1(\mathbb{R}^3))} \le c\|v\|_{L^2(\mathbb{R};(H^1(\mathbb{R}^3))}, \quad \|u\|_{L^2(\mathbb{R};(H^{-1}(\mathbb{R}^3))} \le c\|v\|_{L^2(\mathbb{R};(H^{-1}(\mathbb{R}^3))}.$$

Substituting $u = hv$ into (3.7) and noting that

$$\int_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} u\,dxdt = \int_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} v\,dxdt,$$

we see that the inequality

$$\left|\int_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} v\,dxdt\right| \le cr^{(\gamma-1)/\gamma}\|v\|_{L^2(\mathbb{R};H^1(\mathbb{R}^3))} + cr^{-2}\|v\|_{L^2(\mathbb{R};H^{-1}(\mathbb{R}^3))} \tag{3.8}$$

holds for any function $v \in C_0^\infty(\mathbb{R}^4)$. Therefore, it remains valid for any function $v \in L^2(\mathbb{R}; H^1(\mathbb{R}^3))$. Consider functions $\zeta_k : \mathbb{R}^3 \to \mathbb{R}$, $k \ge 0$, defined by the relations

$$\zeta_0(\xi) = 1 \quad \text{if} \quad |\xi| \le 2; \qquad \zeta_0(\xi) = 0 \quad \text{if} \quad |\xi| > 2,$$

$\zeta_k(\xi) = 1$ if $2^k < |\xi| \le 2^{k+1}$ and $k \ge 1$, and $\zeta_k(\xi) = 0$ otherwise.

The projectors $P_k : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ are defined as

$$\mathcal{F}(P_k u)(\xi) = \zeta_k(\xi)\mathcal{F}u(\xi).$$

Obviously, for any $s \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^3)$, we have

$$c(s)^{-1}\|u\|^2_{H^s(\mathbb{R}^3)} \le \sum_{k\ge 0} 2^{2ks}\|P_k u\|^2_{H^0(\mathbb{R}^3)} \le c(s)\|u\|^2_{H^s(\mathbb{R}^3)}.$$

It follows from this inequality and (3.8) that, for any function $v \in C_0^\infty(\mathbb{R}^4)$,

$$\left|\int_{\mathbb{R}^4} P_k(\rho_\varepsilon^{(\gamma+1)/2}) v\,dxdt\right| = \left|\int_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} P_k v\,dxdt\right| \le cr^{(\gamma-1)/\gamma}\|P_k v\|_{L^2(\mathbb{R};H^1(\mathbb{R}^3))}$$
$$+ cr^{-2}\|P_k v\|_{L^2(\mathbb{R};H^{-1}(\mathbb{R}^3))} \le c(r^{(\gamma-1)/\gamma}2^k + r^{-2}2^{-k})\|P_k v\|_{L^2(\mathbb{R};H^0(\mathbb{R}^3))}.$$

Setting $r = 2^{-k2\gamma/(3\gamma-1)}$ yields the estimate

$$\left|\int_{\mathbb{R}^4} P_k(\rho_\varepsilon^{(\gamma+1)/2}) v\,dxdt\right| \le c2^{k(\gamma+1)/(3\gamma-1)}\|P_k v\|_{L^2(\mathbb{R};H^0(\mathbb{R}^3))}.$$

Since v is arbitrary, it follows that

$$\|P_k(\rho_\varepsilon^{(\gamma+1)/2})\|_{L^2(\mathbb{R}^4)} \le c2^{k(\gamma+1)/(3\gamma-1)}.$$

Noting that $\lambda = (\gamma+1)/(3\gamma-1) + \delta$, $\delta > 0$, we finally obtain the desired estimate (3.1):

$$\|\rho_\varepsilon^{(\gamma+1)/2}\|^2_{L^2(\mathbb{R};H^{-\lambda}(\mathbb{R}^3))} = \sum_k \|P_k(\rho_\varepsilon^{(\gamma+1)/2})\|^2_{L^2(\mathbb{R}^4)} 2^{-2k\lambda} \le c\sum_k 2^{2k(\gamma+1)/(3\gamma-1)-2k\lambda} \le c\sum_k 2^{-2k\delta} \le c.$$

3.2. Proof of Theorem 1.3

According to Lemma 2.2 on localization, to prove the theorem, it is sufficient to show that, for any $\gamma > \gamma^* = (7 + \sqrt{73})/12$, we have

$$\|\rho_\varepsilon|\mathbf{v}_\varepsilon|^2\|_{L^{1+\theta}(\mathbb{R}^4)} \le c, \tag{3.9}$$

where $c > 0$ and $\vartheta > 0$ are independent of ε. To prove (3.9), we fix an arbitrary $N > 0$ and define

$$\mathbf{V}(t) = \mathbf{v}_\varepsilon(t) \quad \text{if} \quad \|\mathbf{v}_\varepsilon(t)\|_{H^1(\mathbb{R}^3)} \le N; \qquad \mathbf{V}(t) = 0 \quad \text{if} \quad \|\mathbf{v}_\varepsilon(t)\|_{H^1(\mathbb{R}^3)} > N.$$

Obviously, the vector field $\mathbf{V}$ is supported by the ball $|x| + |t| \le R$ and satisfies the estimate

$$\|\mathbf{V}\|_{L^\infty(\mathbb{R};H^1(\mathbb{R}^3))} \le N.$$

The subsequent argument relies on the following two lemmas.

Lemma 3.1. *Let* $\lambda \in (0,1)$ *and* $p = 3 - 2\lambda$. *Then, for almost every* $t \in \mathbb{R}$,

$$\left\| |\mathbf{V}(t)|^p \right\|_{H^\lambda(\mathbb{R}^3)} \le c N^p, \tag{3.10}$$

where c *is a constant independent of* ε, t, *or* N.

Proof. For a fixed t, let $W = |\mathbf{V}(t)| N^{-1}$. Obviously,

$$\|W\|_{H^1(\mathbb{R}^3)} \le 1, \qquad |\nabla W^p| \le c W^{p-1} |\nabla W|.$$

Applying the Hölder inequality

$$\int_{\mathbb{R}^3} |\nabla W^p|^q dx \le \left(\int_{\mathbb{R}^3} |W|^{q(p-1)\alpha} dx \right)^{1/\alpha} \left(\int_{\mathbb{R}^3} |\nabla W|^{q\beta} dx \right)^{1/\beta}$$

with exponents

$$\beta = 2/q, \qquad \alpha = 2/(2-q), \qquad q = 6/(2+p),$$

we obtain

$$\int_{\mathbb{R}^3} |\nabla W^p|^q dx \le \left(\int_{\mathbb{R}^3} |W|^6 dx \right)^{1/\alpha} \left(\int_{\mathbb{R}^3} |W|^2 dx \right)^{1/\beta} \le c.$$

Since W^p is supported by the ball $|x| \le R$, it follows that

$$\|W^p\|_{W^{1,q}(\mathbb{R}^3)} \le c.$$

By the embedding theorems for fractional Sobolev spaces (see [10]), the embedding $W^{1,m}(|x| \le R) \to H^\lambda(|x| \le R)$ is continuous for $m = 6/(3 + 2(1-\lambda))$. Setting $q = m$ and noting that $2 + p = 6/q = 5 - 2\lambda$ or $p = 3 - 2\lambda$, we obtain

$$\|W^p\|_{H^\lambda(\mathbb{R}^3)} \le c \|W^p\|_{H^\lambda(|x| \le R)} \le c \|W^p\|_{W^{1,q}(|x| \le R)} \le c$$

for $p = 3 - 2\lambda$. Combining this estimate with $\|W^p\|_{H^\lambda(\mathbb{R}^3)} = N^{-p} \left\| |\mathbf{V}(t)|^p \right\|_{H^\lambda(\mathbb{R}^3)}$ yields the desired estimate (3.10).

Lemma 3.2. *Let* $\gamma > \gamma^*$. *Then there are constants* $\omega > 1$ *and* $c > 0$ *independent of* N, ε *such that*

$$\int_{\mathbb{R}^4} (\rho_\varepsilon |\mathbf{V}|^2)^\omega dx dt \le c N^6. \tag{3.11}$$

Proof. The kinetic energy density is represented as

$$(\rho_\varepsilon |\mathbf{V}|^2)^\omega = |\mathbf{V}|^{(2-k)\omega} (\rho_\varepsilon |\mathbf{V}|^k)^\omega,$$

where the constant k will be specified later. Formally applying the Hölder inequality yields

$$\int_{\mathbb{R}^4} (\rho_\varepsilon |\mathbf{V}|^2)^\omega dx dt \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} \rho_\varepsilon^{(\gamma+1)/2} |\mathbf{V}|^{k(\gamma+1)/2} dx \right)^{2\omega/(\gamma+1)} \left(\int_{\mathbb{R}^3} |\mathbf{V}|^{\omega(2-k)(\gamma+1)/(\gamma+1-2\omega)} dx \right)^{(\gamma+1-2\omega)/(\gamma+1)}. \tag{3.12}$$

The task is to choose parameters k and ω such that the integrals on the right-hand side of the inequality make sense. Let $\lambda \in ((\gamma+1)/(3\gamma-1), 1)$ and

$$k = 2p/(\gamma+1), \qquad p = 3 - 2\lambda.$$

With this choice of k, inequality (3.12) becomes

$$\int\limits_{\mathbb{R}^4} (\rho_\varepsilon |\mathbf{V}|^2)^\omega dxdt \le \int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}^3} \rho_\varepsilon^{(\gamma+1)/2} |\mathbf{V}|\,|^p\, dx \right)^{2\omega/(\gamma+1)} \left(\int\limits_{\mathbb{R}^3} |\mathbf{V}|^\kappa dx \right)^{(\gamma+1-2\omega)/(\gamma+1)} dt,$$

where

$$\kappa = 2\omega(\gamma - 2 + 2\lambda)/(\gamma + 1 - 2\omega).$$

Combining this result with the Young inequality yields the estimate

$$\int\limits_{\mathbb{R}^4} (\rho_\varepsilon |\mathbf{V}|^2)^\omega dxdt \le \int\limits_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} |\mathbf{V}|^p\, dxdt + \int\limits_{R^4} |\mathbf{V}|^\kappa dxdt. \tag{3.13}$$

Since $\lambda \in (0,1)$, inequality (3.1) in Proposition 3.1 and estimate (3.10) in Lemma 3.1 imply the following estimate for the first integral on the right-hand side of (3.13):

$$\int\limits_{\mathbb{R}^4} \rho_\varepsilon^{(\gamma+1)/2} |\mathbf{V}|^p dxdt \le \int\limits_{\mathbb{R}} \left\| \rho_\varepsilon^{(\gamma+1)/2}(t) \right\|_{H^{-\lambda}(\mathbb{R}^3)} dt \left\| |\mathbf{V}|^p \right\|_{L^\infty(\mathbb{R}; H^\lambda(\mathbb{R}^3))} \le cN^p. \tag{3.14}$$

Since $\mathbf{V}(t)$ is supported by the ball $|x| \le R$, the inequality

$$\|\mathbf{V}(t)\|_{L^\kappa(\mathbb{R}^3)} = \|\mathbf{V}(t)\|_{L^\kappa(|x|\le\mathbb{R})} \le c\|\mathbf{V}(t)\|_{H^1(|x|\le\mathbb{R})} = c\|\mathbf{V}(t)\|_{H^1(\mathbb{R}^3)} \le cN \tag{3.15}$$

holds for all $\kappa \in [1,6]$. In turn, the inequality $\kappa \le 6$ holds for all ω close to 1 and for all λ close to $(\gamma+1)/(3\gamma-1)$ if

$$2\left(\gamma - 2 + 2\frac{\gamma+1}{3\gamma-1} \right)/(\gamma - 1) < 6$$

or, equivalently, $6\gamma^2 - 7\gamma - 1 > 0$. It is easy to see that the last inequality holds for all $\gamma > \gamma^*$. Therefore, there exist $\omega > 1$ and $\lambda \in ((\gamma+1)/(3\gamma-1),\, 1)$ such that $\kappa \le 6$ and

$$\int\limits_{\mathbb{R}^4} |\mathbf{V}|^\kappa dxdt \le cR\|\mathbf{V}\|^\kappa_{L^\infty(\mathbb{R};L^\kappa(\mathbb{R}^3))} \le cR\|\mathbf{V}\|^\kappa_{L^\infty(\mathbb{R};H^1(\mathbb{R}^3))} = cN^\kappa. \tag{3.16}$$

Here, we used the fact that $\mathbf{V}(t) = 0$ for $|t| \ge R$. Combining estimates (3.13), (3.14), and (3.16) and noting that $p \le 3$ and $\kappa \le 6$, we derive the desired estimate (3.11).

Now, the proof of the main estimate (3.9) can be completed. Fix arbitrary $K, N > 1$ and introduce the sets

$$\mathscr{E}_N = \left\{ t : \|\mathbf{v}_\varepsilon(t)\|_{H^1(\mathbb{R}^3)} \le N \right\},$$

$$\mathscr{A}_N = \mathbb{R}^3 \times \mathscr{E}_N, \quad \mathscr{B}_N = \mathbb{R}^3 \times (\mathbb{R} \backslash \mathscr{E}_N),$$

$$\mathscr{C}_K = \left\{ (x,t) : \rho_\varepsilon(x,t)|\mathbf{v}_\varepsilon(x,t)|^2 \ge K \right\}, \quad \mathscr{C}_K(t) = \left\{ x : \rho_\varepsilon(x,t)|\mathbf{v}_\varepsilon(x,t)|^2 \ge K \right\}.$$

The definition of $\mathbf{V}(t)$ and inequality (3.11) imply the estimate

$$\int\limits_{\mathscr{A}_N} (\rho_\varepsilon |\mathbf{v}_\varepsilon|^2)^\omega dxdt = \int\limits_{\mathscr{A}_N} (\rho_\varepsilon |\mathbf{V}|^2)^\omega dxdt \le cN^6.$$

It follows that, for any $K > 1$,

$$\operatorname{meas}(\mathscr{C}_K \cap \mathscr{A}_N) \le cK^{-\omega} N^6, \tag{3.17}$$

where c is a constant independent of ε, K, or N. Note that the energy estimate implies that

$$\int\limits_{\mathbb{R}^3} \rho_\varepsilon(t)|\mathbf{v}_\varepsilon(t)|^2\, dx \le c,$$

which yields the estimate

$$\operatorname{meas} \mathscr{C}_K(t) \le cK^{-1}. \tag{3.18}$$

On the other hand, the estimate $\|\mathbf{v}_\varepsilon\|_{L^2(\mathbb{R},H^1(\mathbb{R}^3))} \le c$ and the definition of the set $\mathscr{E}_N$ imply

$$\text{meas}(\mathbb{R}\backslash\mathscr{E}_N) \le cN^{-2}. \tag{3.19}$$

Combining (3.18) with (3.19), we obtain the estimate

$$\text{meas}(\mathscr{C}_K \cap \mathscr{B}_N) = \int_{\mathbb{R}\backslash\mathscr{E}_N} \text{meas}\,\mathscr{C}_K(t)dt \le cK^{-1}N^{-2},$$

which, together with (3.17), yields the inequality

$$\text{meas}\,\mathscr{C}_K = \text{meas}(\mathscr{C}_K \cap \mathscr{A}_N) + \text{meas}(\mathscr{C}_K \cap \mathscr{B}_N) \le c(K^{-\omega}N^6 + K^{-1}N^{-2}).$$

Setting $N = K^{(\omega-1)/8}$, we derive

$$\text{meas}\left\{(x,t) : \rho_\varepsilon|\mathbf{v}_\varepsilon|^2 \ge K\right\} = \text{meas}\,\mathscr{C}_K \le cK^{-1-(\omega-1)/4},$$

where K is an arbitrary number greater than 1. It follows that the desired estimate (3.9), namely,

$$\left\|\rho_\varepsilon|\mathbf{v}_\varepsilon|^2\right\|_{L^{1+\vartheta}(\mathbb{R}^4)} \le c$$

holds for any $\vartheta \in (0,(\omega-1)/4)$.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation, project no. 15-11-20019.

REFERENCES

1. P. L. Lions, *Mathematical Topics in Fluid Dynamics*, Vol. 2: *Compressible Models* (Clarendon, Oxford, 1998).
2. E. Feireisl, A. Novotný, and H. Petzeltova, "On the existence of globally defined weak solutions to the Navier–Stokes equations of compressible isentropic fluids," J. Math. Fluid Mech. **3** (3), 358–392 (2001).
3. E. Feireisl, *Dynamics of Viscous Compressible Fluids* (Oxford Univ. Press, Oxford, 2004).
4. A. Novotný and I. Straškraba, *Introduction to the Mathematical Theory of Compressible Flow*, Oxford Lecture Series in Mathematics and Its Applications, Vol. 27 (Oxford Univ. Press, Oxford, 2004).
5. P. I. Plotnikov and J. Sokolowski, *Compressible Navier–Stokes Equations: Theory and Shape Optimization* (Birkhäuser, Basel, 2012).
6. S. Jiang and P. Zhang, "On spherically symmetric solutions of the compressible isentropic Navier–Stokes equations," Commun. Math. Phys. **215**, 559–581 (2001).
7. B. Ducomet, S. Nečasová, and A. Vasseur, "On spherically symmetric motions of a viscous compressible barotropic and self-gravitating gas," J. Math. Fluid. Mech. **99**, 1–24 (2009).
8. S. Jiang and P. Zhang, "Axisymmetric solutions to the 3D Navier–Stokes equations for compressible isentropic fluids," J. Math. Pures Appl. **82**, 949–973 (2003).
9. W. Sun, S. Jiang, and Z. Guo, "Helically symmetric solutions to the 3D Navier–Stokes equations for compressible isentropic fluids," J. Differ. Equations **222**, 263–296 (2006).
10. R. A. Adams, *Sobolev Spaces* (Academic, New York, 1975).

Translated by I. Ruzanova